Forces on bins: The effect of random friction

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The q model of Coppersmith *et al.* [Phys. Rev. E **53**, 4673 (1996)] has renewed interest in understanding the forces generated along the walls and at the bottom of a silo filled with a granular material. Fluctuations in the mean stress have been characterized for the q model, and related to experimental work on stress chains. The classical engineering approach to bin loads follows from Janssen's analysis [Z. Ver. Dtsch. Ing. **39**, 1045 (1895)], predicting a saturation of stress, as a function of depth, in a tall silo. In this paper we reexamine the Janssen theory, introducing randomness into the important parameters in the theory. The Janssen analysis relies on assumptions not met in practice. For this reason, we numerically solve the partial differential equations governing the equilibrium of forces in a bin, again including randomness in parameters. We show that the most important of these parameters is a coefficient of friction at the wall of the bin. This random friction model combines some features of fluctuations as seen in experiments, with a classical continuum mechanics approach to describing granular materials. [S1063-651X(98)03003-7]

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I. INTRODUCTION

The classical engineering theory of Janssen [1] provides an estimate for the mean vertical stress in a silo filled with a granular material. The principal feature of the Janssen analysis is that, under passive stress conditions, the mean stress saturates, asymptoting to a value depending on bin radius and wall and internal friction coefficients, but independent of the height. The Janssen theory relies on two assumptions, assumptions which do not hold in practice. Nevertheless, the analysis gives a reasonable estimate of the bin loads, and its simplicity is its virtue. Several analyses have attempted to remove some of the assumptions of the Janssen theory; the interested reader should consult Ref. [2].

Recently Coppersmith and co-workers [3,4] developed a model for the force distribution in a bin. In this ''q model,'' any one particle within the sample transmits its weight to neighbors that are below it, in a random manner. The authors derived a mean field theory based on this model, and found fluctuations in the forces felt by the lowest row of particles. Under most assumptions on the choice of random number distribution, the number of occurrences of a fluctuation of a given size decays exponentially with size.

The current incarnation of the q model is scalar: only the vertical force is balanced. Recently, Socolar [5] introduced a generalization, the so-called α model, which balances vertical and horizontal forces and angular moment. The α model contains three random variables, and analysis appears difficult. However numerical simulations modeling rough walled bins appear consistent with the classical continuum theory; numerical simulations modeling infinitely wide bins appear consistent with the q model for a special distribution of the q's.

To provide a framework for introducing fluctuations into a continuum setting, we incorporate some of the randomness of the q model into the Janssen analysis. In Sec. II we reconsider the Janssen derivation, include a random component into the grain friction, and reformulate the balance law as a stochastic differential equation. Standard results of stochastic calculus provide an estimate of the mean stress and its variance, at any height. In Sec. III, we numerically solve the complete stress equilibrium equations, assuming a Mohr-Coulomb constitutive relation, and again including a random component in the friction. Under passive loading, the stress saturates; stress fluctuations are not significant until near saturation.

An experimental finding closely related to the current note is Ref. [6]. That paper reported careful measurements of force fluctuations in tall narrow bins, bins whose widths range from 3–8 grain diameters and whose depth ranges up to about 100 grain diameters. Measured average vertical stress at any depth is systematically higher than predicted by the Jansen theory, and fluctuations in this stress range up to about 20%. These fluctuations are apparent only after the stress starts to saturate. (Socolar [5] also found the Janssen stress to be smaller than his calculated average stress, at any depth.) These experiments also demonstrated a dependence of stress on ambient temperature, an effect we do not consider here.

II. GENERALIZED JANSSEN ANALYSIS

We briefly review Janssen's theory, and provide a stochastic generalization of that analysis. See Ref. [2] for the fundamental mechanics of granular media. All of this study is restricted to two space dimensions.

Let the average vertical stress be denoted $\overline{\sigma} = \int_{-D/2}^{D/2} \sigma^{yy}(x,y)\, dx$, where σ^{xx}, σ^{xy}, and σ^{yy} are the xx, xy, and yy components, respectively, of the (symmetric) stress tensor T.

Consider the force diagram in Fig. 1; at equilibrium, the average stress at y and $y+\Delta y$, gravity, and wall friction $\overline{\tau}$ are balanced:

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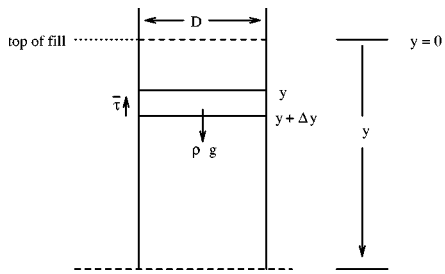

FIG. 1. Force balance for Janssen's analysis. On the slice, stresses and gravity are balanced by wall friction.

$$\partial_y \overline{\sigma} + \frac{2\,\overline{\tau}}{D} = \rho g. \tag{1}$$

Now we make two assumptions, critical to the Janssen theory, but which do not hold in practice.

(1) At every point σ^{xx} and σ^{yy} are the principal stresses (i.e., the eigenvalues of the stress tensor) and the Coulomb frictional condition implies that $\sigma^{xx}(x,y) = K\sigma^{yy}(x,y)$, $K = (1+s)/(1-s)$, and $s = \sin(\phi)$, and ϕ is the internal friction angle.

(2) Along the wall, $\overline{\tau} = \sigma^{xy}(\pm D/2,y) = \delta\sigma^{xx}(\pm D/2,y)$ where $\delta = \tan(\phi_w)$, ϕ_w is the wall-material friction angle.

Combining these assumptions, we arrive at the equation

$$\partial_y \overline{\sigma} + \alpha\sigma = \rho g, \quad \alpha = \frac{2\,\delta K}{D}. \tag{2}$$

Solving subject to $\overline{\sigma} \to 0$, as $y \to 0$, gives

$$\overline{\sigma}(y) = \frac{\rho g}{\alpha}[1 - \exp(-\alpha y)]. \tag{3}$$

It is apparent that the average stress saturates, the asymptotic value $\rho g/\alpha$ depending on the material and wall parameters and the bin diameter.

The formula for K is based on the assumption that the stress field is in the passive state, with the xx stress the major principal stress (the larger of the eigenvalues) and the yy stress the minor (the smaller eigenvalue). If the material is in the active state, the yy stress is major, the xx stress minor, and K is replaced by K^{-1}. For a typical material, ϕ may be 30°, so $K = 3$ in the passive state. In the active state this parameter is $\frac{1}{3}$, and saturation of the stress requires a bin that is an order of magnitude taller.

Now assume that the coefficient of the stress in Eq. (2) has both mean and fluctuating components. This fluctuating component might arise from randomness in the friction angle, for example. Assuming an Itô formulation for the resulting stochastic differential equation, write

$$d\overline{\sigma} = -\alpha\overline{\sigma}\,dy - \epsilon\overline{\sigma}\,dW + \rho g\,dy. \tag{4}$$

Here $dW(y)$ is a Wiener measure associated with the random fluctuations, and ϵ is a measure of the size of the fluctuations. Standard arguments give the following results (see, e.g., Chap. 8 in Ref. [7]). A formal solution may be obtained by a variation of parameters argument, but more insightful are formulas for the first and second moments. The mean of the solution, $m \doteq \mathcal{E}(\overline{\sigma})$, is, not surprisingly, the Janssen solution (3). The second moment $P \doteq \mathcal{E}(\overline{\sigma^2})$, satisfies

$$\dot{P} = (-2\alpha + \epsilon^2)P + 2m\rho g.$$

Thus

$$P = \frac{2m\rho g}{2\alpha - \epsilon^2}\{1 - \exp[-(2\alpha - \epsilon^2)y]\}. \tag{5}$$

The standard deviation is $\sqrt{P - m^2}$, and an order of magnitude estimate gives the deviation $\sim m\epsilon/\sqrt{2\alpha}$, after the stress has saturated.

An alternative hypothesis is that randomness in packing leads to fluctuations in the density, and thus to fluctuations in the stress. That is, the weight ρg must include a random component due to voids. This assumption leads to the equation

$$d\overline{\sigma} = -\alpha\overline{\sigma}\,dy + \rho g\,dy + \epsilon\rho g\,dW. \tag{6}$$

The mean of the solution is, again, given by Eq. (3). The standard deviation is $(\epsilon\rho g)/\sqrt{2\alpha}[1 - \exp(-2\alpha y)]^{1/2}$.

III. EQUILIBRIUM ANALYSIS

The Janssen analysis relies on assumptions not met in practice. In this section we solve the full stress equilibrium equations for a Coulomb material in a bin. Although analysis is possible in the limiting case of smooth walls (see Ref. [2]), this section determines solutions numerically.

The stress equilibrium is written

$$\partial_x \sigma_{xx} + \partial_y \sigma_{xy} = 0, \tag{7}$$

$$\partial_x \sigma_{yx} + \partial_y \sigma_{yy} = \rho g. \tag{8}$$

A common constitutive assumption is that the material is Mohr-Coulomb, at incipient yield. That is, one assumes the ratio of the shear stress, τ, to the mean stress, σ, is a constant, where

$$\sigma = \frac{\sigma_1 + \sigma_2}{2}, \quad \tau = \frac{\sigma_1 - \sigma_2}{2}, \tag{9}$$

and σ_1 and σ_2 are the eigenvalues of the stress tensor T. The Mohr-Coulomb condition reads

$$\frac{\tau}{\sigma} = s. \tag{10}$$

The Mohr-Coulomb condition can be viewed as a nonlinear relation for, say, σ_{yy} in terms of σ_{xx} and σ_{xy}. It is often convenient to make a change of variables that incorporates this relation. With the mean stress σ defined above, introduce the angle ψ, measured from the horizontal, such that $[\cos(\psi), \sin(\psi)]$ is an eigenvector of T associated with σ_1. Then write

$$T = \sigma\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sigma s\begin{pmatrix} \cos(2\psi) & \sin(2\psi) \\ \sin(2\psi) & -\cos(2\psi) \end{pmatrix}.$$

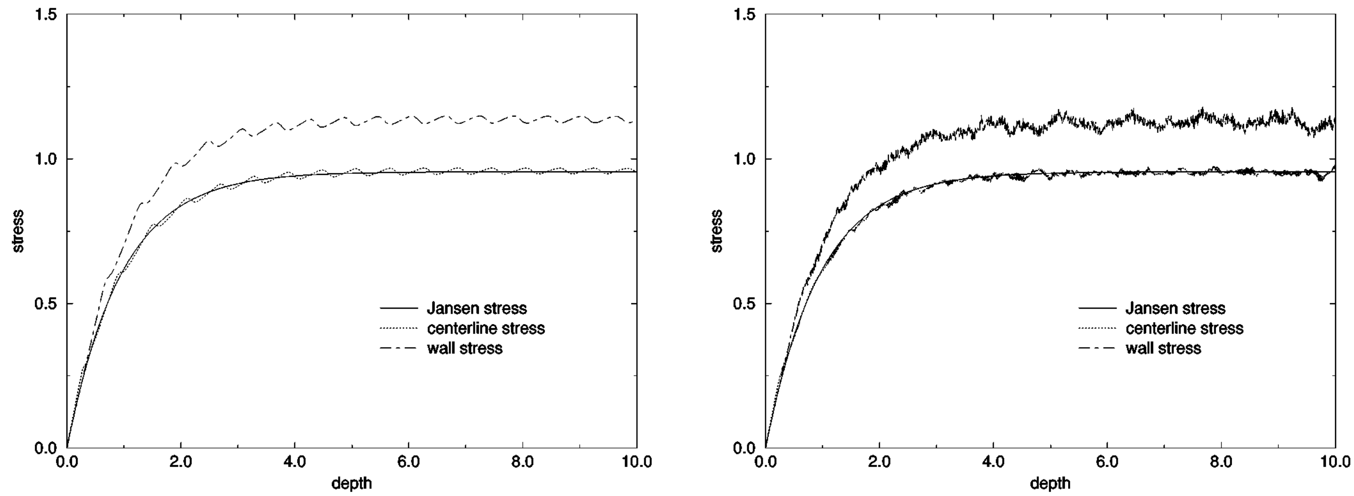

FIG. 2. (a) The yy component of the stress at the centerline and the wall, with no random component of friction. For comparison the Janssen solution is also plotted. Here the nominal internal friction angle $\phi = 30°$ and the nominal wall friction angle $\delta = 15°$. (b) Similar plot, but with a random component added to the friction angles.

This equation specifies the stresses in terms of two dependent variables σ and ψ, whose evolution is determined by the equilibrium equations.

This change of variables may be used to rewrite the momentum equations as

$$\begin{pmatrix} 1+s\cos(2\psi) & -2\sigma s\sin(2\psi) \\ s\sin(2\psi) & 2\sigma s\cos(2\psi) \end{pmatrix} \partial_x \begin{pmatrix} \sigma \\ \psi \end{pmatrix} + \begin{pmatrix} s\sin(2\psi) & 2\sigma s\cos(2\psi) \\ 1-s\cos(2\psi) & 2\sigma s\sin(2\psi) \end{pmatrix} \partial_y \begin{pmatrix} \sigma \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \rho g \end{pmatrix}.$$

We nondimensionalize by scaling length by the bin diameter D, and stress by $\rho g D$. All calculations are reported in nondimensional units. The independent variable are $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$ and $0 \leqslant y \leqslant H$. This system of partial differential equations is strictly hyperbolic, with characteristics inclined at an angle $\pm[(\pi/4)-(\phi/2)]$ from the direction of major principal stress. The y direction may be taken as the timelike direction. ''Initial'' conditions for σ and ψ are imposed at the top of the fill, $y=0$, and the equations are solved downward. At the boundaries, the bin walls $x=\pm\frac{1}{2}$, the wall friction angle is imposed: $\psi=\delta$. The system of equations is solved by a modification of the total variation diminishing (TVD)/central difference scheme of Nessyahu and Tadmor [8]. The method is second order accurate, and designed to avoid spurious oscillations common to many higher-order schemes for hyperbolic systems. For the computations reported here, a grid size of $\Delta x = 0.02$ was used. On very coarse grids, fluctuations are larger than shown; after sufficient refinement, the size of fluctuations appears to stabilize.

To introduce fluctuations, at each gridpoint at each level, the friction angle is chosen with a random component. Specifically, if ϕ is the nominal friction angle, the angle used is $\phi_{\mathrm{fluct}} = \phi(1.0+\zeta\xi)$, where ξ is chosen randomly from a uniform distribution between $[-0.5, 0.5]$, and ζ is an adjustable parameter measuring the extent of variation in the friction angle. Depending on testing apparatus, variations in measurements of the internal friction angle are as large as $\pm 5°$, more than 10% of typical values [9]; without good measurements of the span of friction angles, we conservatively set $\zeta = 0.1$. So, for a nominal friction angle of 30°, $\phi_{\mathrm{fluct}} \in [28.5, 31.5]$; the sine of this angle is used in the constitutive relation, and this sine $\in [0.477, 0.522]$, about a $\pm 5\%$ swing. Of course variations of friction in a real sample may have spatial correlations; absent good modeling justification for a particular choice of correlation, none is used here. A random component of the wall friction angle (the boundary condition δ) is added in a manner similar to ϕ. We emphasize that our choice of ζ sets the imposed variation in the friction angle, and thus of the stress, but this choice is rather arbitrary.

The first result to understand is a typical stress profile, without any friction fluctuation, and the same parameters but with fluctuation. This is shown in Fig. 2, which displays the yy component of the stress at the centerline of the bin and at the bin wall. For comparison, the Janssen stress is also shown. We have imposed the ''initial condition'' $\sigma = 0$ on $y=0$. However, the condition for a surface $y=h(x)$ to be stress free is (in general) inconsistent with $y=$const; impos-

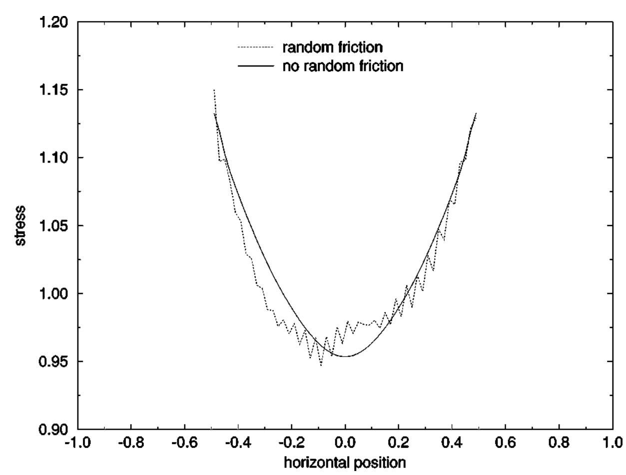

FIG. 3. Variation in the yy stress across the width of the bin, at $y=10$. Shown are results both with and without a random component of the friction angles. In both cases, the nominal friction angles are $\phi = 30°$ and $\delta = 15°$.

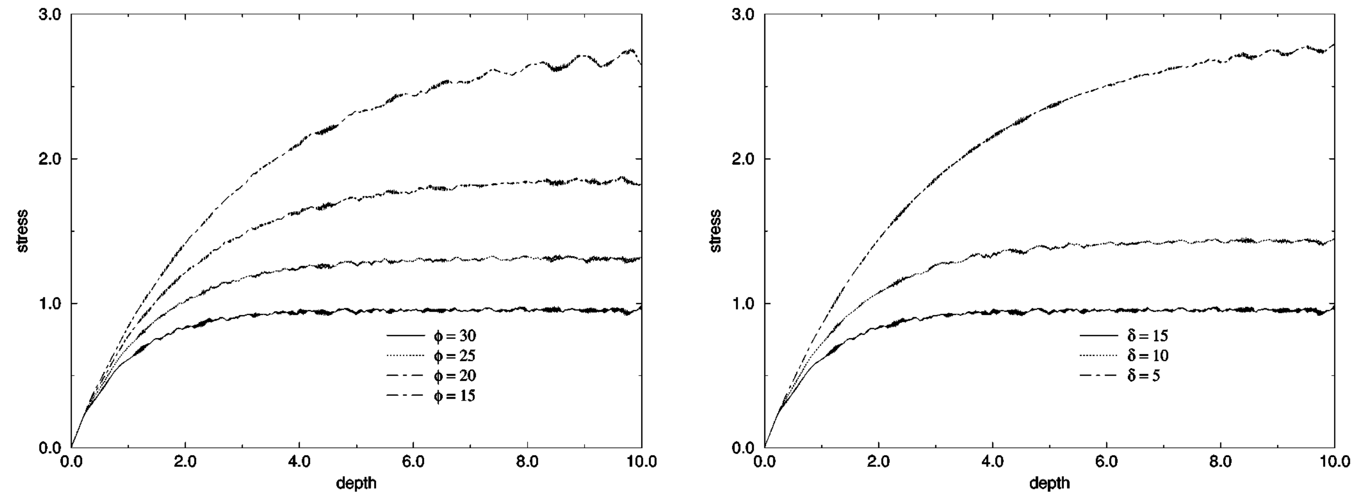

FIG. 4. (a) The yy component of the stress at the centerline for different nominal internal friction angles. The nominal wall friction is held fixed $\delta = 15°$. (b) The yy component of the stress at the centerline for different nominal wall friction angles, with a fixed nominal internal friction angle $\phi = 30°$.

ing $\sigma = 0$ leads to a free boundary problem for the upper surface, a problem we do not wish to address here. The regular oscillations in Fig. 2(a) are due to mismatch in the imposed stress at the intersection of the $y = 0$ surface and the bin wall, and are well documented (see, e.g., references in Ref. [2]); the period of these oscillations is related to the speed of the characteristics of the hyperbolic system. In Fig. 2(b), fluctuations at the walls are larger than at the centerline, and the wall stress is some 15% larger than at the centerline. Notice that the regular oscillations in Fig. 2(a) are dissipated by the randomness.

In Fig. 3, the yy stress is shown as a function of position across the bin, at the depth $y = 10$, the terminus of the computations in Fig. 2. The variation across the bin illustrates the limitations of the Janssen assumptions. Nonetheless, Figs. 2 and 3 show that the Janssen analysis provides a good estimate of the centerline stress (and *not* of the average stress). This partially explains why the measurements of Ref. [6] are larger than the Janssen predictions. The centerline stress is typically 15–20 % smaller than the largest stresses, found at the wall.

Figure 4 illustrates the sensitivity of computations to changes in the nominal friction angles. In Fig. 4(a), the wall friction is held fixed, while the nominal internal friction angle ϕ is varied from 15° to 30° (recall that the random fluctuation is 5% of the nominal angle). With lower internal friction, fluctuations become more pronounced. We conjecture that this is due to a lower friction angle transmitting a smaller fraction of stress (and of stress fluctuations) to the walls, leaving a larger fraction of stress (and of stress fluctuations) to be transmitted vertically. Notice too that, at the smallest friction angle, the regular oscillations of the stress reappear. When internal friction is held constant but wall friction is varied [Fig. 4(b)], the stress saturates deeper in the bin, and fluctuations are not apparent until after this saturation. We note that, with no wall friction, no weight is transferred to the bin walls and a hydrostatic stress results. Similarly, when periodic boundary conditions are imposed, a hydrostatic stress results.

In Fig. 5, fluctuations for two sets of friction angles are plotted. In each case, the equations were solved through $y = 50$. The centerline stress for $20 < y < 50$ was extracted, and the average computed; this average should be the asymptotic value of the stress. A normalized deviation from the mean was found by subtracting the mean from the sample value, and dividing by the mean. For viewing, one signal is offset by 0.05. For the baseline case $\phi = 30°$ and $\delta = 15°$, the internal friction angle varies by about $\pm 1.5°$, and the wall friction angle by about $\pm 0.75°$; the stress exhibits fluctuations of about $\pm 4\%$. For the cases $\phi = 30°$ and $\delta = 5°$, the internal friction angle again varies by about $\pm 1.5°$, but the wall friction varies by only $\pm 0.25°$; the stress fluctuates about $\pm 2.5\%$.

Figure 6 provides a plot of spectral power for the baseline cases $\phi = 30°$ and $\delta = 15°$. The stress was computed to a depth of $y = 50$; recall from Fig. 2 that, for the given friction angles, the stress saturates well before $y = 10$. The centerline stress is sampled at every second time step, from about $y = 20$ to 50. The power is computed using a Welch window with overlap, on the last 2560 sampled values. Shown is the

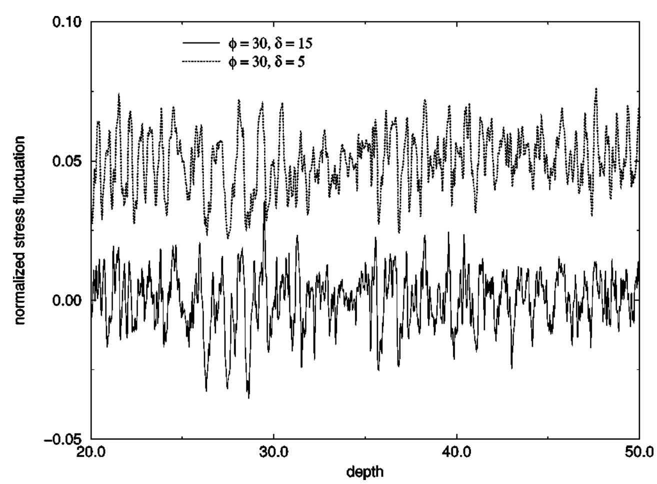

FIG. 5. Normalized fluctuations in the centerline yy stress for two pair of friction angles. Both signals are demeaned; for viewing, the top signal is vertically offset by 0.05.

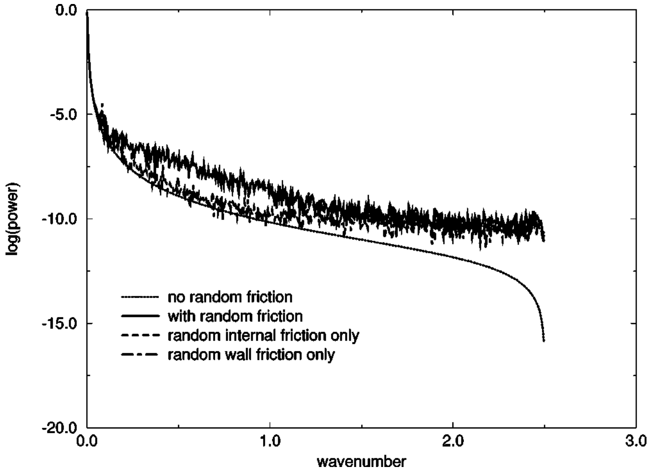

FIG. 6. Log (base 10) of the spectral power for four variations of the base case. The variations are no random component of friction, a random component of both internal and wall friction, a random component added to internal friction only, and a random component added to wall friction only. The nominal $\phi=30°$ and $\delta=15°$.

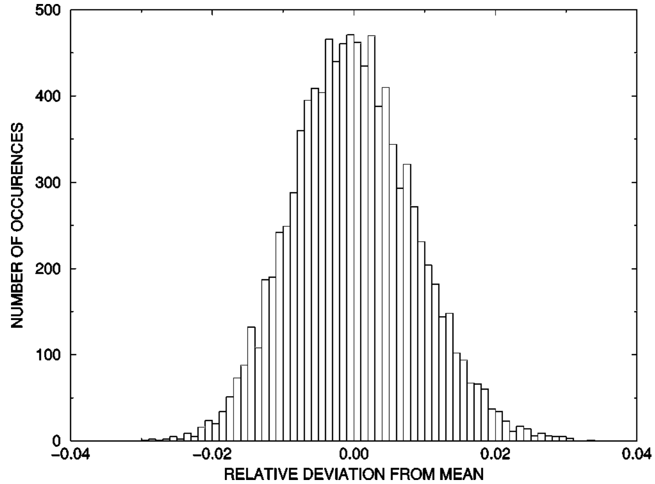

FIG. 7. A histogram of the normalized deviation of the centerline yy stress from the mean, evaluated at $y=50$. Friction parameters were $\phi=30°$ and $\delta=15°$. Computed for 10 000 realizations, the distribution appears Gaussian, not exponential, as predicted by the q model.

(base 10 log of the) power for four variations: (i) no random component of friction, (ii) a random component of both internal and wall friction, (iii) a random component added to internal friction only, (iv) a random component added to wall friction only. The power for wall friction only lies atop the spectrum for wall and internal friction. The power for internal friction only deviates from these at lower wave numbers. Thus fluctuations in the stress are essentially due to a random component in the wall friction angle. From Fig. 5, these fluctuations range up to about $\pm 4\%$ of the mean. Recall that this variation is based on about a 5% variation in the friction coefficient. The fluctuations reported in Ref. [6] are as large as 20%. This comparison suggests that a 15–20 % variation in the friction coefficient is not an unreasonable parameter in stochastic models like the present.

Analysis of the q model shows that the number of occurrences of a fluctuation of a given size decays exponentially with size. Recent experiments [10] on short bins verify this finding, for stresses larger than the mean; stresses smaller than the mean decay like a power law. Figure 7 presents the distribution for the random friction model. The equilibrium equations were solved to a depth $y=50$, and the centerline stress was recorded. Ten thousand realizations were made. The average over all realizations was calculated, subtracted from the sample value, and this difference was normalized by the average. Figure 7 is a histogram of these relative deviations. The distribution of fluctuations appears Gaussian, not exponential.

IV. SUMMARY

We reexamined the Janssen analysis incorporating a random component of friction, solving for the mean and the second moment of the stress. For comparison, the nonlinear equilibrium equations for a Mohr-Coulomb material with random friction were solved numerically. The analysis suggests that fluctuations are significant only after the stress begins to saturate, a finding consistent with the experimental work of Ref. [6]. The primary contribution to stress fluctuations is randomness in the wall friction, a boundary condition. The fluctuations found in this model are set by a free parameter defining the magnitude of random friction. Our choice of this parameter results in fluctuations of about 5% of the mean stress, much less than the 15–20 % found in experiments.

The results presented here are in qualitative agreement with those of Socolar's α model. His calculations incorporated stress balance in both horizontal and vertical directions, and a balance of angular momentum. The essential feature of the α model is that particle friction transmits stress from particle to particle and, ultimately, to the walls of a bin. These stresses, and any stress fluctuation, are partially absorbed by the wall. In contrast, the q model only considers vertical forces; stresses predicted by the q model are more like hydrostatic forces, and there is no mechanism for dissipating fluctuations.

A difficulty faced by all of these models is correlations. Experiments [11] show chains of particles experiencing high stress (the frequency of which falls off exponentially with size). These pictures, and many other experiments, suggest that grain forces are correlated. However, we lack adequate information to introduce correlations into models in a meaningful way. Experimental results reported Ref. [10] measure static forces on short bins, and show no evidence of correlations. The question of whether there are correlations, and over what length scales are they important, is central to the entire formulation of a continuum framework for granular materials. Experimental and theoretical work is necessary to understand the nature of correlations.

Mueth, Jaeger, and Nagel [10] also studied the frequency of fluctuations in three dimensional systems. They found that, for fluctuations larger than the mean, the frequency of fluctuation of a given size decays exponentially with size of fluctuations. For fluctuations smaller than the mean, the decay follows a power law. Furthermore, their findings are largely unaffected by changes in the boundary friction. For purposes of comparison with this work, several factors are important. The experimental setup has a depth-width aspect ratio of about 1–1.5. The glass beads and acrylic used in the experiment are very low friction materials, with both internal and wall friction angles about $10°-15°$. From the continuum perspective, stresses measured in this arrangement are hydrostaticlike. Walls do not support the bead pack, and even moderate changes in the wall friction would have only minor effects on stress measurements. We do not view these findings as invalidating the random friction model proposed here, at least not for engineering applications. The random packing model offers one possible explanation for these experimental findings. The mean stress for this model is given by Eq. (3) in the limit $\alpha \to 0$, and equals $\rho g y$; the standard deviation, Eq. (5) in the $\alpha \to 0$ limit, is $\epsilon \rho g \sqrt{y}$. For a short bin, $y \approx 1$, and fluctuations are on the order of ϵ times the mean stress. Packing variations, interpreted as voids fraction, can range up to 20–30 %. However, even this model does not explain all the physics of small aspect ratio bins.

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